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A CAUCHY PROBLEM FOR AN ELLIPTIC EQUATION IN A STRIP

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Abstract: Let $p \in (1, \infty), \varphi \in L_p(\mathbb{R})$ and $\varepsilon < E$ be given nonnegative constants. In this paper we prove stability estimates of Hölder type for the Cauchy problem

$$u_{xx} + a(y)u_{yy} + b(y)u_y + c(y)u = 0, \quad -\infty < x < \infty, \quad 0 < y < 1,$$
$$\|u(\cdot, 0) - \varphi\|_p \le \varepsilon,$$
$$u_y(x, 0) = 0, \quad -\infty < x < \infty.$$

subject to the constrain $\|u(\cdot, 1)\|_p \le E$. Furthermore, we suggest a marching difference scheme for solving the problem in a stable way.

1. INTRODUCTION

Let $p \in (1, \infty), \varphi \in L_p(\mathbb{R})$ and ε, M be given constants such that $0 \le \varepsilon < M < \infty$. In this paper we consider the Cauchy problem

$$u_{xx} + a(y)u_{yy} + b(y)u_y + c(y)u = 0, \quad -\infty < x < \infty, \quad 0 < y < 1, \tag{1}$$
$$\|u(\cdot, 0) - \varphi\|_p \le \varepsilon, \tag{2}$$
$$u_y(x, 0) = 0, \quad -\infty < x < \infty, \tag{3}$$

subject to the constraint

$$\|u(\cdot, 1)\|_p \le E. \tag{4}$$

Here a, b, c are given functions such that for some given positive constants $\lambda \le A$

$$\lambda \le a(y) \le A, \qquad y \in [0, 1] \tag{5}$$
$$a(y) \in C^2[0, 1], \quad b(y) \in C^1[0, 1], \quad c(y) \in C[0, 1], \quad c(y) \le 0. \tag{6}$$

Without loss of generality in the following we suppose that $\lambda \ge 1$.

We note that the problem with the non-homogeneous right-hand side in (1) and the second Cauchy data in (3) can be transformed to the above problem via an auxiliary well-posed boundary problem. When $\varepsilon = 0$ we shall prove that a solution of (1)–(3) (see Definition below) exists if and only if φ is infinitely differentiable and $\|\varphi^{(n)}\|_{L_p(\mathbb{R})} \le cn!s^n$, $\forall n \in \mathbb{N}$ for some nonnegative constants c and s. The Cauchy problem (1)-(3) is well-known to be ill posed: a small perturbation in the Cauchy data may cause a very large error in the solution (see, e.g. [5, 8, 9, 11]). It is therefore difficult to develop numerical methods for the problem, since we always have errors of measurement in the Cauchy data, discretization errors and round-off errors; all of these make numerical solutions unstable. To overcome this difficulty we shall apply the mollification method of [3] in order to solve the problem (1)-(3) in a stable way and prove some stability estimates of Hölder type for the solution and its derivatives. The idea of the method is as follows: we mollify the Cauchy data φ by the convolution with the de la Vallée Poussin kernel (with the Dirichlet kernel for $p = 2$). The mollified data belong to the space of band-limited functions, in which the Cauchy problem is well-posed. And with appropriate choices of mollification parameters we obtain errors estimates of Hölder type. Further, since the Fourier transform of band-limited functions has compact support, the method can be easily implemented numerically using the fast Fourier transform (FFT) technique. Based on this remark we suggest in Section 6 a stable marching difference scheme for (1)-(3). This paper is organized as follows. In Section 2 we outline some results on the Cauchy problem in frequency space. Section 3 is devoted to stability estimates for the L_2 case and Sections 4 and 5 for the $L_p(1 < p \le \infty)$ cases. Finally, a stable marching difference scheme in the L_p-norm for (1)-(3) is presented in Section 6. We separately analyze the L_2 and the general L_p cases since the techniques for them are totally different. Furthermore, the L_2 case is much easier and for which we get much more sharp estimates than for the general L_p ones. We note that stability estimates in the L_p-norm of Hölder type for ill-posed problems are not so developed. Furthermore, stable marching difference schemes in the L_p-norm are quite interesting not only for ill-posed problems but also for well-posed ones. This paper is a further development of [4] and also supplies some corrections to it.

In this paper we shall make use of the following notation: $\mathfrak{M}_{\nu,p}$ $(1 \le p \le \infty)$ will denote the collection of all entire functions of exponential type ν which as functions of a real $x \in \mathbb{R}$ lie in $L_p = L_p(\mathbb{R})$ ([14, p. 100]). We shall denote by $E_{\nu,p}(f)$ the best approximation of f using elements of $\mathfrak{M}_{\nu,p}$ ([14, p. 184]) $E_{\nu,p}(f) = \inf_{g\in\mathfrak{M}_{\nu,p}} \|f-g\|_{L_p(\mathbb{R})}$. The norm in L_p will be denoted by $\|\cdot\|_p$. For a function $f \in L_1(\mathbb{R})$, its Fourier transform is defined by $F[f](\xi) = \hat{f}(\xi) = 1/\sqrt{2\pi} \int\limits_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$. The inverse Fourier transform is denoted by F^{-1}. In the paper $c_1, c_2, \dots$ are generic positive constants.

2. AUXILIARY RESULTS

Consider the Cauchy problem in frequency space

$$-z^2v(z,y) + a(y)v_{yy}(z,y) + b(y)v_y(z,y) + c(y)v(z,y) = 0, \quad 0<y<1, \quad z\in\mathbb{C}, \tag{7}$$

$$v(z,0) = 1, \quad z\in\mathbb{C}, \tag{8}$$

$$v_y(z,0) = 0, \quad z\in\mathbb{C}. \tag{9}$$

Since the properties of $v(z,y)$ play an essential role in obtaining stability results for the Cauchy problem (1)-(3), we study them carefully and then derive some direct consequences.

Lemma 1. *There exists a unique solution of* (7)-(8) *such that*

i) $v(z,\cdot) \in W^{2,\infty}(0,1), \;\; \forall z\in\mathbb{C}$,

ii) $v(z,y)$ *is an entire function of* z *for every* $y \in [0,1]$,

iii) $v(z,1) \neq 0, \;\; \forall z \neq ci, c\in\mathbb{R}, i = \sqrt{-1}$,

iv) there exist constants c_1, c_2, c_3 *and* c_4 *such that for* $z \in \mathbb{R}$

$$|v(z,y)| \le c_1 e^{|z|A(y)}, \qquad \forall y\in[0,1], \tag{10}$$

$$|v(z,1)| \ge c_2 e^{|z|A(1)}, \tag{11}$$

and if $|z| \ge z_0 > 0$*, then* $\forall y \in [0,1]$,

$$|v_y(z,y)| \le c_3|z|e^{|z|A(y)} \text{ and } |v_{yy}(z,y)| \le c_4|z|^2e^{|z|A(y)}.$$

Here $A(y) := \int_0^y \frac{ds}{\sqrt{a(s)}}, \quad y\in[0,1]$.

We use the technique of Knabner and Vessella [10] with some modifications to prove this lemma.

Lemma 2. *Let* $k(z,y)$ *be the solution of the boundary value problem*

$$a(y)k_{yy}(z,y) + b(y)k_y(z,y) + c(y)k(z,y) = z^2k(z,y), \quad 0<y<1, \quad z\in\mathbb{C}, \tag{12}$$

$$k_y(z,0) = 0, \quad z\in\mathbb{C}, \tag{13}$$

$$k(z,1) = 1, \quad z\in\mathbb{C}. \tag{14}$$

Then, there exists a constant c_5 *such that*

$$|k(z,y)|, |k_z(z,y)|, |k_{zz}(z,t)| \le c_5 e^{(A(y)-A(1))|z|} \tag{15}$$

and for any $\alpha \in (0,1)$,

$$|z||k_z(z,y)| \le \frac{c_5}{1-\alpha} e^{\alpha(A(y)-A(1))|z|}. \tag{16}$$

To prove these results we use lemma 1 and [6,7].

Proposition 1. *Let $p \in (1,\infty)$ and the function $\psi \in L_p(\mathbb{R})$. Consider the boundary value problem*

$$u_{xx} + a(y)u_{yy} + b(y)u_y + c(y)u = 0, \quad -\infty < x < \infty, \quad 0 < y < 1, \tag{17}$$

$$u_y(x,0) = 0, \quad -\infty < x < \infty, \tag{18}$$

$$u(x,1) = \psi(x), \quad -\infty < x < \infty. \tag{19}$$

Then for any fixed $y \in [0,1)$, the solution $u(x,y)$ is analytic in the variable x and it can be analytically extended in the strip $\mathbb{R}\times i(-\sigma(y),\sigma(y))$ of the complex plane $\mathbb{C}$ with $\sigma(y) \in (0, A(1)-A(y))$. Furthermore, there is a constant c_6 such that

$$\|u(\cdot \pm i\sigma(y), y)\|_p \le c_6 \frac{A(1)-A(y)}{A(1)-A(y)-\sigma(y)}\|\psi\|_p. \tag{20}$$

The proof of this proposition is based on lemmas 1, 2 and the following result of Mikhlin [13] (see also [14, chapter 1]): Denote by $\mathbb{R}^n_* = \mathbb{R}^n\backslash\{0\}$. Let $k \in C^n(\mathbb{R}^n_*)$. If for a multi-index α there is a constant c such that

$$|\xi|^{|\alpha|}|D^\alpha k(\xi)| \le c, \quad \xi \in \mathbb{R}^n_*, \quad |\alpha| \le n,$$

then the mapping $\psi \mapsto F^{-1}kF\psi$ is continuous in $L_p(\mathbb{R}^n)$ and there is a constant $K(n,p)$ such that

$$\|F^{-1}[kF\psi]\|_{L_p(\mathbb{R}^n)} \le K(n,p)c\|\psi\|_{L_p(\mathbb{R}^n)}, \quad \forall\psi \in L_p(\mathbb{R}^n).$$

Proposition 2. *Let $u(x,y)$ be defined as in Proposition 1. Then there is a constant c_7 such that*

$$E_{\nu,p}(u(\cdot,y)) \le c_7 \frac{A(1)-A(y)}{A(1)-A(y)-\sigma(y)}\|\psi\|_p e^{-\sigma(y)\nu}.$$

In particular, if we take $\sigma(y) = \alpha(A(1)-A(y))$ with arbitrary $\alpha \in (0,1)$, then

$$E_{\nu,p}(u(\cdot,y)) \le c_7 \frac{1}{1-\alpha}\|\psi\|_p e^{-\alpha(A(1)-A(y))\nu}.$$

To prove this proposition we need the following remarkable result of Bernstein (see [1,15]).

Lemma 3. *For $\sigma > 0$, let $f(x+iy)$ be analytic in the strip $\mathbb{R}\times i(-\sigma,\sigma)$ and real valued for $y = 0$. If there is a constant m such that $\|Re\ f(\cdot \pm i\sigma)\|_p \le m, 1 \le p \le \infty$, then we have*

$$E_{\nu,p}(f(\cdot)) \le c_8 m e^{-\sigma\nu}.$$

Here, c_8 is a defined constant.

The statement of Proposition 2 follows now from this lemma and Proposition 1.

Proposition 3. *Let $u(x,y)$ be defined as in Proposition 1. Then there is a constant c_9 such that with arbitrary $\alpha \in (0,1)$ the following inequalities hold for $n = 1,2,\ldots$*

$$E_{\nu,p}\left(\frac{\partial^n}{\partial x^n}u(\cdot,y)\right) \le c_9 \frac{(n+1)!}{(1-\alpha)^{n+1}(A(1)-A(y))^{n+1}}\|\psi\|_p e^{-\alpha(A(1)-A(y))\nu}.$$

To deal with the case $p = \infty$ we need the following result.

Lemma 4. *For any $y \in [0,1)$, the function*

$$F^{-1}[k](x \pm i\sigma(y), y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k(\xi,y)e^{i(x\pm i\sigma(y))\xi}d\xi$$

is well defined for all $\sigma(y) \in [0, A(1)-A(y))$. It is analytic with respect to the variable $z = x+i\sigma(y)$ in the strip $\mathbb{R}\times i(A(y)-A(1), A(1)-A(y))$ of the complex plane $\mathbb{C}$. Furthermore, $F^{-1}[k](\cdot \pm i\sigma(y), y) \in L_1(\mathbb{R})$ and

$$\|F^{-1}[k](\cdot \pm i\sigma(y))\|_1 \le \frac{c_9}{A(1)-A(y)-\sigma(y)} \tag{21}$$

Lemma 5. *Let $u(x,y)$ be the solution of the boundary value problem (17)-(19) with $\psi \in L_\infty(\mathbb{R})$. Let further that $y \in [0,1)$ and $|\sigma| < A(1) - A(y)$. Then, there is a constant c_{10} such that, for any $\alpha \in (0,1)$,*

$$E_{\nu,p}(u(\cdot,y)) \leq \frac{c_{10}}{(1-\alpha)(A(1)-A(y))} e^{-\alpha(A(1)-A(y))\nu} \|\psi\|_\infty.$$

Using the results in this section we can prove the following results in §3-5 on stability estimates.

2. $\mathbf{L_2}$−CASE

2.1. Solvability

Definition 2.1. We say that $u(x,y)$ is an L_2-solution of (1)-(4) if for any $y_0 \in (0,1]$, $u(\cdot,y_0) \in L_2(\mathbb{R})$ and u satisfies (1)-(4).

When $\varepsilon = 0$, if in the problem (1)-(3), $u(\cdot,y) \in L_2(\mathbb{R})$ exists for $y < 1$, we say that the solution is local.

Theorem 1. *If there exists an L_2-solution of (1)–(4), then $u(x,0)$ is infinitely differentiable and there exist some constants c and s such that*

$$\|u^{(k)}(x,0)\|_2 \leq ck!s^k, \quad \forall k = 0,1,2,\ldots \tag{22}$$

Conversely, if a function φ satisfies (22), then there exists a local L_2-solution of the system

$$\begin{aligned}
&u_{xx} + a(y)u_{yy} + b(y)u_y + c(y)u = 0, \quad -\infty < x < \infty, \quad 0 < y < 1,\\
&\|u(\cdot,0) - \varphi\|_2 = 0,\\
&u_y(x,0) = 0, \quad -\infty < x < \infty.
\end{aligned}$$

2.2. Mollification method

To solve (1)–(4) in a stable way we mollify φ by the convolution with the Dirichlet kernel

$$\varphi \to \varphi^\nu(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \frac{\sin \nu(x-\eta)}{x-\eta} d\eta \tag{23}$$

for some positive ν, and instead of considering (1)–(3) with φ we look for its mollified version

$$u^\nu_{xx} + a(y)u^\nu_{yy} + b(y)u^\nu_y + c(y)u^\nu = 0, \quad -\infty < x < \infty, \quad 0 < y < 1, \tag{24}$$

$$u^\nu(x,0) = \varphi^\nu(x), \quad -\infty < x < \infty, \tag{25}$$

$$u^\nu_y(x,0) = 0, \qquad -\infty < x < \infty. \tag{26}$$

Theorem 2. *For any fixed $\nu > 0$ the problem (24)–(26) is solvable and its solution is stable*

$$\left\| \frac{\partial^m u^\nu(\cdot,y)}{\partial x^m} \right\| \leq c_1 \nu^m e^{\nu A(y)} \|\varphi\|, \quad 0 \leq y \leq 1, \quad m = 0,1,2,\ldots \tag{27}$$

Furthermore, for ε small enough, with

$$\nu = \nu^* = \frac{1}{A(1)} \ln \frac{E}{\varepsilon} \tag{28}$$

for $y \in (0,1)$ we have

$$\|u(\cdot,y) - u^{\nu^*}(\cdot,y)\| \leq c_1 \Big(1 + \frac{1}{c_2}\Big) \varepsilon^{\frac{A(y)}{A(1)}} E^{1-\frac{A(y)}{A(1)}} \tag{29}$$

and

$$\left\| \frac{\partial^m u(\cdot,y)}{\partial x^m} - \frac{\partial^m u^{\nu^*}(\cdot,y)}{\partial x^m} \right\| \leq c_1 \Big(1 + \frac{1}{c_2} \big(\frac{m}{A(1)-A(y)}\big)^m\Big) \Big(\frac{1}{A(1)} \ln \frac{E}{\varepsilon}\Big)^m \varepsilon^{\frac{A(y)}{A(1)}} E^{1-\frac{A(y)}{A(1)}}, \quad m = 1,2,\ldots \tag{30}$$

Here, $u(x,y)$ is a solution of (1)–(4) and $u^{\nu^*}(x,y)$ is the solution of (24)–(26) with $\nu = \nu^*$.

2.3. Stability estimates
From theorem 2 and the triangle inequality we immediately obtain the following result.

Theorem 3. *Let $u_1(x,y), u_2(x,y)$ be any solutions of the problem (1)-(4). Then,*

$$\|u_1(\cdot,y)-u_2(\cdot,y)\| \le 2c_1\Big(1+\frac{1}{c_2}\Big)\varepsilon^{\frac{A(y)}{A(1)}}E^{1-\frac{A(y)}{A(1)}} \tag{31}$$

and, for $m=1,2,...,$

$$\Big\|\frac{\partial^m u_1(\cdot,y)}{\partial x^m}-\frac{\partial^m u_2(\cdot,y)}{\partial x^m}\Big\| \le 2c_1\Big(1+\frac{1}{c_2}(\frac{m}{A(1)-A(y)})^m\Big)\Big(\frac{1}{A(1)}\ln\frac{E}{\varepsilon}\Big)^m\varepsilon^{\frac{A(y)}{A(1)}}E^{1-\frac{A(y)}{A(1)}}. \tag{32}$$

In the case of Laplace's equation we can also obtain stability estimates for derivatives of u with respect to y.

Theorem 4. *Let $u_1(x,y), u_2(x,y)$ be L_2-solutions of the Cauchy problem for Laplace's equation*

$$\begin{aligned} &u_{xx}+u_{yy}=0, \quad -\infty<x<\infty, \quad 0<y<1,\\ &\|u(\cdot,0)-\varphi\|_2\le\varepsilon, \quad -\infty<x<\infty,\\ &u_y(x,0)=0, \quad -\infty<x<\infty, \end{aligned}$$

with $\varphi\in L_2(\mathbb{R})$, $\|u(\cdot,1)\|_2\le E$ and $\varepsilon\le E$. Then for any $m=0,1,2,\dots$, $\ell=0,1,2,\dots$ $m+\ell\ge1$, $y\in[0,1)$, we have

$$\Big\|\frac{\partial^{m+\ell}u_1(\cdot,y)}{\partial x^m\partial y^\ell}-\frac{\partial^{m+\ell}u_2(\cdot,y)}{\partial x^m\partial y^\ell}\Big\| \le 2\Big(\ln\Big(e\frac{E}{\varepsilon}\Big)\Big)^{m+\ell}\Big(e^y+2(\frac{m+\ell}{1-y})^{m+\ell}\Big)\varepsilon^{1-y}E^y.$$

4. $\mathbf{L_p(1<p<\infty)}$ CASE

Theorem 5. *For $y\in[0,1)$, a solution $u(x,y)$ of the problem (1)–(4) is analytic with respect to the variable x. Further, for any $\alpha\in(0,1-y)$,*

$$\Big\|\frac{\partial^n u(\cdot,y)}{\partial x^n}\Big\|_p \le c_2\frac{(n+1)!}{(1-\alpha)^{n+1}(A(1)-A(y))^{n+1}}E, \quad n=0,1,2,\dots$$

In particular, when $y=0$, for any $\alpha\in(0,1)$,

$$\|\frac{\partial^n u(\cdot,0)}{\partial x^n}\|_p \le c_2\frac{(n+1)!}{(1-\alpha)^{n+1}A(1)^{n+1}}E. \tag{33}$$

Conversely, if $\bar\varphi=u(x,0)$ satisfies (33), then there is a local solution in L_p of the Cauchy problem

$$\begin{aligned} &u_{xx}+a(y)u_{yy}+b(y)u_y+c(y)u=0, \quad -\infty<x<\infty, \quad 0<y<1,\\ &u(x,0)=\bar\varphi(x), \quad -\infty<x<\infty,\\ &u_y(x,0)=0, \quad -\infty<x<\infty. \end{aligned}$$

Now we return to the "generalized" Cauchy problem (1)–(4). We shall prove some stability estimates for this problem by the mollification method of [3].

Denote the de la Vallée Poussin kernel [14, p. 304] by

$$k_\nu(x)=\frac{1}{\pi\nu}\frac{\cos(\nu x)-\cos(2\nu x)}{\nu^2}\ , \quad \nu>0.$$

This kernel belongs to $\mathfrak{M}_{2\nu,1}$ and has many nice properties. In particular, the convolution of a function $\varphi\in L_p(\mathbb{R})$ with this kernel belongs to $\mathfrak{M}_{2\nu,p}$ ([14, p. 304–306])

$$\varphi^\nu(x):=\int_{-\infty}^{\infty}k_\nu(x-z)\varphi(z)dz\in\mathfrak{M}_{2\nu,p}.$$

Now we mollify φ by the convolution with the de la Vallée Poussin kernel and consider the mollified Cauchy problem

$$u^{\nu}_{xx} + a(y)u^{\nu}_{yy} + b(y)u^{\nu}_{y} + c(y)u^{\nu} = 0, \quad -\infty < x < \infty,\ 0 < y < 1, \tag{34}$$
$$u^{\nu}(x,0) = \varphi^{\nu}(x), \quad -\infty < x < \infty, \tag{35}$$
$$u^{\nu}_{y}(x,0) = 0, \quad -\infty < x < \infty. \tag{36}$$

Theorem 6. *There exists a unique solution of the mollified Cauchy problem (34)–(36). For any $y > 0$, $u^{\nu}(\cdot,y) \in \mathfrak{M}_{2\nu,p}$ and for any y finite it is stable in the L_p-norm. Further, for any $\alpha \in (0,1)$ and $y \in [0,1]$, with*

$$\nu = \nu(y) = \frac{1}{2A(y) + \alpha(A(1) - A(y))} ln\frac{E}{\varepsilon}$$

we have

$$\|u^{\nu}(\cdot,y) - u(\cdot,y)\|_p \le \left(2\sqrt{3}c_1 + (1+2\sqrt{3})\frac{c_7}{1-\alpha}\right) E^{\frac{2A(y)}{2A(y)+\alpha(A(1)-A(y))}} \varepsilon^{1-\frac{2A(y)}{2A(y)+\alpha(A(1)-A(y))}}. \tag{37}$$

Proof. We note that $\ln\frac{E}{\varepsilon} > 0$, since $\varepsilon < E$. Therefore $\nu(y) > 0$, and so the choice for ν is acceptable. Since $\varphi^{\nu} \in \mathfrak{M}_{2\nu,p}$ and the function $v(z,y)$ is analytic in z, the existence and uniqueness of a solution $u^{\nu}(\cdot,y) \in \mathfrak{M}_{2\nu,p}$ for any $y > 0$ is guaranteed by a general theory on the Cauchy problems ([16]).

From lemmas 1, 2, the Bernstein-Nikolskii inequality[1] for the functions in $\mathfrak{M}_{2\nu,p}$, and Gronwall's inequality, we get

$$\|u^{\nu}(\cdot,y)\|_p \le c_1 e^{2\nu A(y)} \|\varphi^{\nu}\|_p. \tag{38}$$

Here, c_1 is the same constant as in proposition 1. Thus, u^{ν} is stable in the L_p-norm for fixed $\nu > 0$.

To estimate $\|u^{\nu} - u\|_p$ we use $u(x,y) = v(\cdot,y) * u(\cdot,1)$ and set $\overline{\varphi}(x) := u(x,0) = F^{-1}[k](\cdot,0) * u(\cdot,1)$.

Then $\overline{\varphi} \in L_p(\mathbb{R})$ and $\|\varphi - \overline{\varphi}\|_p \le \varepsilon$. The Cauchy problem

$$\overline{u}^{\nu}_{xx} + a(y)\overline{u}^{\nu}_{yy} + b(y)\overline{u}^{\nu}_{y} + c(y)\overline{u}^{\nu} = 0, \quad -\infty < x < \infty,\ 0 < y < 1,$$
$$\overline{u}^{\nu}(x,0) = \overline{\varphi}^{\nu}(x), \quad -\infty < x < \infty,$$
$$\overline{u}^{;\nu}_{y}(x,0) = 0, \quad -\infty < x < \infty$$

with $\overline{\varphi}^{\nu}$ being the convolution of $\overline{\varphi}$ with k_{ν} has a unique solution in $\mathfrak{M}_{2\nu,p}$. In virtue of Bernstein-Nikolskii's inequality and the properties of the de la Valée Poussin kernel,

$$\left\|u^{\nu}(\cdot,y) - \overline{u}^{\nu}(\cdot,y)\right\|_p \le 2\sqrt{3}c_1 e^{2A(y)\nu}\varepsilon.$$

On the other hand, we can prove that $\overline{u}^{\nu}(x,y) = k_{\nu}(\cdot) * u(\cdot,y)$. Consequently, by the property of the de la Vallée Poussin kernel and Lemma 2,

$$\|\overline{u}^{\nu}(\cdot,y) - u(\cdot,y)\|_p \le (1+2\sqrt{3})\frac{c_7}{1-\alpha} E e^{-\alpha(1-A(y))\nu}$$

for $y \in [0,1)$ and $\alpha \in (0,1)$.

Finally, for $y \in [0,1)$ and $\alpha \in (0,1)$,

$$\begin{aligned} \|u^{\nu}(\cdot,y) - u(\cdot,y)\|_p &\le \|u^{\nu}(\cdot,y) - \overline{u}^{\nu}(\cdot,y)\|_p + \|\overline{u}^{\nu}(\cdot,y) - u(\cdot,y)\|_p \\ &\le 2\sqrt{3}c_1 e^{2A(y)\nu}\varepsilon + (1+2\sqrt{3})\frac{c_7}{1-\alpha} E e^{-\alpha(A(1)-A(y))\nu}. \end{aligned}$$

The inequality (37) for $y \in [0,1)$ now follows directly from this estimate. The case $y = 1$ is trivial.

The following result is an immediate corollary of the previous theorem and the triangle inequality.

[1] Bernstein-Nikolskii's inequality says that if a function f belongs to $\mathfrak{M}_{\nu p}$, then $\|f^{(n)}\|_p \le \nu^n \|f\|_p, \forall n = 1, 2, \ldots$ ([14, p. 116])

Theorem 7. *Let u_1 and u_2 be any solutions of (1)–(4). Then, for any $\alpha \in (0,1)$ and $y \in [0,1]$,*

$$\|u_1(\cdot,y) - u_2(\cdot,y)\|_p \le 2\left(2\sqrt{3}c_1 + (1+2\sqrt{3})\frac{c_7}{1-\alpha}\right) E^{\frac{2A(y)}{2A(y)+\alpha(A(1)-A(y))}} \varepsilon^{1-\frac{2A(y)}{2A(y)+\alpha(A(1)-A(y))}}.$$

Remark 1. If in Theorem 6 we take ν independent of y, say

$$\nu = \nu^* = \frac{1}{2A(1)}\ln\frac{E}{\varepsilon}, \tag{39}$$

then we have a slightly weaker estimate

$$\begin{aligned}\|u^{\varepsilon,\nu^*}(\cdot,y) - u(\cdot,y)\|_p &\le 2\sqrt{3}c_1 E^{A(y)/A(1)} \cdot \varepsilon^{1-A(y)/A(1)} \\ &\quad + (1+2\sqrt{3})\frac{c_7}{1-\alpha} E^{1-\alpha(A(1)-A(y))/2A(1)} \varepsilon^{\alpha(A(1)-A(y))/2A(1)}.\end{aligned}$$

However, this choice is convenient for numerical implementations.

Remark 2. To have stability estimates at $y = 1$ we have to impose some more regularity conditions on $u(x,1)$ (see [4]).

Theorem 8. *Let u_1 and u_2 be any solutions of (1)–(4). Then, for any $\alpha \in (0,1), y \in [0,1)$, $n = 0,1,2,\ldots$, there is a constant c_{11} such that*

$$\begin{aligned}\left\|\frac{\partial^n}{\partial x^n}u_1(\cdot,y) - \frac{\partial^n}{\partial x^n}u_2(\cdot,y)\right\|_p &\le c_{11}\left(\frac{1}{A(1)^n}\left(\ln\frac{E}{\varepsilon}\right)^n E^{\frac{A(y)}{A(1)}} \varepsilon^{1-\frac{A(y)}{A(1)}}\right. \\ &\quad \left. + \frac{(n+1)!}{(1-\alpha)^{n+1}(A(1)-A(y))^{n+1}} E^{1-\alpha\frac{A(1)-A(y)}{2A(1)}} \varepsilon^{\alpha\frac{A(1)-A(y)}{2A(1)}}\right).\end{aligned}$$

For the Laplace equation we can get also stability estimates for the derivatives with respect to x and y.

5. L_∞ CASE

Proceeding as in the previous section, from lemma 5, for any $\alpha \in (0,1)$, we have

$$\|u^\nu(\cdot,y) - u(\cdot,y)\|_\infty \le c_1 e^{2\nu A(y)} + \frac{c_{10}}{(1-\alpha)(A(1)-A(x))} e^{-\alpha(A(1)-A(y))\nu} E.$$

Thus, if we choose

$$\nu = \frac{1}{\alpha A(1) + (2-\alpha)A(y)} \ln\frac{E}{\varepsilon},$$

then

$$\|u^\nu(\cdot,y) - u(\cdot,y)\|_\infty \le \left(c_1 + \frac{c_{10}}{(1-\alpha)(A(1)-A(x))}\right) E^{\frac{2A(y)}{\alpha A(1)+(2-\alpha)A(y)}} \varepsilon^{1-\frac{2A(y)}{\alpha A(1)+(2-\alpha)A(y)}}.$$

Hence for any solutions $u_1(x,y), u_2(x,y)$ of (1)–(4) with $p = \infty$ we have the stability estimate

$$\|u_1(\cdot,y) - u_2(\cdot,y)\|_\infty \le 2\left(c_1 + \frac{c_{10}}{(1-\alpha)(A(1)-A(x))}\right) E^{\frac{2A(y)}{\alpha A(1)+(2-\alpha)A(y)}} \varepsilon^{1-\frac{2A(y)}{\alpha A(1)+(2-\alpha)A(y)}}.$$

Although for any fixed $y \in [0,1)$ this estimate is of Hölder type, it blows up when y tends to 1. Thus, the estimate in the L_∞-case is unfortunately local. However, remark 2 is still valid for this case.

6. STABLE MARCHING DIFFERENCE SCHEME

Since the symbol $v(\xi,y)$ is not always found exactly, it makes the mollification method sometimes not directly applicable. In this section we suggest a stable marching difference scheme based on the mollification method for the Cauchy problem (1)–(4) with noisy data φ. To do this we first mollify φ with the mollification parameter ν according to (39), then Theorem 6 says that our mollified problem is stable and we have error estimates of Hölder type as indicated there. For simplicity, set

$$U := u^{\varepsilon,\nu}, \qquad W := u_y^{\varepsilon,\nu}, \qquad \Psi := \varphi^{\varepsilon,\nu}. \tag{40}$$

Then we have the system of first-order differential equations

$$U_y = W, \quad x \in \mathbb{R}, \quad y \in (0,1), \tag{41}$$
$$a(y)W_y + b(y)W + c(y)U + U_{xx} = 0, \quad x \in \mathbb{R}, \quad y \in (0,1), \tag{42}$$
$$U(x,0) = \Psi, \qquad x \in \mathbb{R}, \tag{43}$$
$$W(x,0) = 0, \qquad x \in \mathbb{R}. \tag{44}$$

We introduce the uniform grid on $\mathbb{R} \times [0,1]$ plane

$$\big\{x_n = nh, \quad y_k = k\tau \big| n = 0, \pm 1, \pm 2, \dots, \ k = 0, 1, \dots, N, \ N\tau = 1\big\}.$$

For a function $f(x,y)$ defined on $\mathbb{R} \times [0,1]$ set $f_n^k = f(nh, k\tau)$.

We discretize (41)-(44) as follows

$$\frac{U_n^{m+1} - U_n^m}{\tau} = W_n^{m+1}, \quad n = 0, \pm 1, \dots, m = 0, 1, \dots, N-1, \tag{45}$$
$$a^m \frac{W_n^{m+1} - W_n^m}{\tau} + b^m W_n^m + c^m U_n^m + \frac{U_{n+1}^m - 2U_n^m + U_{m-1}^n}{h^2} = 0,$$
$$n = 0, \pm 1, \dots \ m = 0, 1, \dots, N-1, \tag{46}$$
$$U_n^0 = \Psi_n, \quad n = 0, \pm 1, \dots \tag{47}$$
$$W_n^0 = 0, \quad n = 0, \pm 1, \dots \tag{48}$$

This system is in fact a marching difference scheme:

$$U_n^0 = \Psi_n, \quad n = 0, \pm 1, \dots \tag{49}$$
$$W_n^0 = 0, \quad n = 0, \pm 1, \dots \tag{50}$$
$$W_n^{m+1} = W_n^m - \tau \frac{b^m}{a^m} W_n^m - \tau \frac{c^m}{a^m} U_n^m - \frac{\tau}{a^m} \frac{U_{n+1}^m - 2U_n^m + U_{n-1}^m}{h^2}$$
$$n = 0, 1, \dots \ m = 0, 1, \dots, N-1 \tag{51}$$
$$U_n^{m+1} = U_n^m + \tau W_n^{m+1}. \tag{52}$$

Theorem 9. *The difference scheme (45)–(48) approximates the problem (41)–(44) with a truncation error which behaves like $O(h^2 + \tau^2)$. Furthermore, if $h \le \pi/\nu$, then it is unconditionally stable.*

Proof. The first assertion is clear. We prove only the stability of the scheme. In doing so we need the notion of the discrete Fourier transform. Suppose that the sequence $f_h := \{f_j\}_{j=0}^{\infty} \in \ell_p, 1 < p < \infty$. It means that

$$\|\{f_j\}\|_{\ell_p} := \left(\sum_{j=-\infty}^{\infty} |f_j|^p \right)^{1/p} < \infty.$$

We define for f_h its discrete Fourier transform as follows

$$\overset{\Delta}{f_h}(\omega) = \frac{h}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} f_j e^{-i\omega jh}, \quad -\frac{\pi}{h} \le \omega \le \frac{\pi}{h}.$$

Lemma 6. *(Marcinkiwicz' theorem) Let $1 < p < \infty$. Then for any $f \in \mathfrak{M}_{\nu,p}$, there are two constants C_1 and C_2 such that*

$$C_1 \nu^{1/p} \|f\|p \le \|\{f(n\frac{\pi}{\nu})\}\|_p \le C_2 \nu^{1/p} \|f\|_p.$$

A proof of this lemma can be found in [2] or [12, p. 152].

Corollary. *Let $1 < p < \infty$ and $0 < h \le \nu$. Then for any $\{f_j\} \in \ell_p$, the series*

$$\sum_{j=-\infty}^{\infty} f_{jh} sinc\Big(\frac{\pi}{h}(x - jh)\Big)$$

converges in the L_p-norm and converges uniformly in any compact of $\mathbb{R}$ to a function $f \in \mathfrak{M}_{\nu,p}$, which is the unique solution of the interpolation problem $f(jh) = f_j, j \in \mathbb{Z}$. Here, sinc$(x) = \sin(x)/x$ for $x \neq 0$ and $= 1$ for $x = 0$.

Based on this result, we associate any $\{f_j\}_{j=-\infty}^{\infty} \in \ell_p$ with the function $f \in \mathfrak{M}_{\nu,p}$ defined by the series

$$f(x) := \sum_{j=-\infty}^{\infty} f_{jh}\text{sinc}\left(\frac{\pi}{h}(x - jh)\right).$$

Proposition 4. *Let $1 < p < \infty, h \leq \pi/\nu$ and $f \in \mathfrak{M}_{\nu,p}$. Then $\hat{f}(\omega) = \overset{\Delta}{f_h}(\omega), \quad |\omega| \leq \pi/h$.*

Lemma 7. *Let $1 \leq p \leq \infty, h \leq \pi/\nu$ and $g \in \mathfrak{M}_{\nu,p}$. Then $\left\| F\left[4\frac{\sin^2\frac{\xi h}{2}}{h^2}\hat{g}(\xi)\right]\right\|_p \leq \frac{5}{3}\nu^2\|g\|_p$.*

Now we are in a position to prove the remain part of the theorem. Note that from the mollification method either $\Psi = \varphi^\nu$ belongs to $\mathfrak{M}_{\nu,p}$ when $p = 2$, or $\mathfrak{M}_{2\nu,p}$ for the general cases $1 < p < \infty$. For simplicity we write $\Psi \in \mathfrak{M}_{\nu,p}$ for both cases.

Since $\Psi \in \mathfrak{M}_{\nu,p}$, from proposition 4 we have supp $\overset{\Delta}{U^0}(\omega) \subset [-\nu, \nu]$. It follows that

$$\text{supp } \overset{\Delta}{U^1}(\omega), \quad \text{supp } \overset{\Delta}{W^1}(\omega) \subset [-\nu, \nu]$$

and so

$$\text{supp } \overset{\Delta}{U^m}(\omega), \quad \text{supp } \overset{\Delta}{W^m}(\omega) \subset [-\nu, \nu], \quad m = 0, 1, \ldots, N-1.$$

We associate the series $\left\{\frac{U_{n+1}^m - 2U_n^m + U_{m-1}^n}{h^2}\right\}_{j=-\infty}^{\infty}$ with the function $\Delta_h U^m$. Its discrete Fourier transform is $4\left(\frac{\sin^2\frac{\omega h}{2}}{h^2}\right)\overset{\Delta}{U^m}$. Since supp $\overset{\Delta}{U^m} \subset [-\nu, \nu]$, from proposition 4 and lemma 7, we have

$$\|\{\Delta_h U^m\}\|_{\ell_p} \leq \frac{5}{3}\frac{C_2}{C_1}\nu^{1/p}\nu^2\|\{U^m\}\|_{\ell_p}. \tag{53}$$

Since $\Psi \in \mathfrak{M}_{\nu,p}$, from (49)–(50) and lemma 6, we have

$$\|\{U_n^0\}\|_{\ell_p} = \|\{\Psi_n\}\|_{\ell_p} \leq C_2\nu^{1/p}\|\Psi\|_p \text{ and } \|\{W_n^0\}\|_{\ell_p} = 0.$$

Let $|b| \leq B$, $|c| \leq C$. Since supp $\overset{\Delta}{U^m} \subset [-\nu, \nu]$, the interpolated functions U^m and W^m belong to $\mathfrak{M}_{\nu,p}$. Hence, from (51), (52), (53) and the inequality $\tau = \frac{1}{N} \leq 1$, recurrently we can prove that

$$\max\{\|\{U^{m+1}\}\|_{\ell_p}, \|\{W^{m+1}\}\|_{\ell_p}\} \leq \exp\left(1 + \frac{B + C}{\lambda} + \frac{5}{3}\frac{C_2}{C_1}\nu^{2+1/p}C_2\right)\nu^{1/p}\|\Psi\|_p.$$

Thus, our scheme is unconditionally stable. The theorem is proved.

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